Comparison of Gross & Harris and Jain CDF Formulas

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2005 December 2

In *Optimizing Oracle Performance* [Millsap & Holt (2003): O'Reilly], I noted an incorrectness in the Jain CDF Formula (p236). Frank Ives has kindly provided work showing that the inclusion of a single pair of parenthesis makes the Jain formula equivalent to the Gross & Harris formula. This notebook is a further validation that the Gross & Harris formula and the Jain formula, with Ives's modification, indeed have identical values for all input values, as long as $C = A$ (which is just the "stable queueing system" assumption).

Note also that neither the Jain formula nor the Gross & Harris formula is strictly correct unless you also add the following qualifying condition:

$$\rho \neq \frac{m-1}{m}.$$

Without this qualifying condition, both formulas have a singularity. For example, neither formula can produce a value for $m = 4$, $\rho = .75$. This singularity *is* mentioned in the book. The VisualBasic code for the CDFr function shown on page 235 uses a special conditional block to interpolate the CDF value at the singularity from the two domain values $\rho = \frac{m-1}{m} \pm \epsilon$.

```
Clear["Global`*"];
λ := a / t;
τ := 1 / λ;
x := c / t;
s := b / c;
μ := 1 / s;
u := b / t ;
ρ := u / m;
c := a ; (* ...on a stable queueing system. *)
```

$$\text{erlangc} := \frac{\frac{(m\,\rho)^m}{m!}}{(1-\rho)\sum_{k=0}^{m-1}\frac{(m\,\rho)^k}{k!} + \frac{(m\,\rho)^m}{m!}};$$

$$\text{p0} := \left(\sum_{n=0}^{m-1}\frac{(m\,\rho)^n}{n!} + \frac{(m\,\rho)^m}{m!\,(1-\rho)}\right)^{-1};$$

$$\text{Wq0} := 1 - \frac{(m\,\rho)^m\,\text{p0}}{m!\,(1-\rho)};$$

$$\mathbf{jain} = 1 - e^{-\mu r} - \frac{\mathbf{erlangc}}{1 - m + m\rho}\left(e^{-m\mu(1-\rho) r} - e^{-\mu r}\right);$$

$$\mathbf{gross} = \frac{m(1-\rho) - \mathrm{Wq0}}{m(1-\rho) - 1}\left(1 - e^{-\mu r}\right) - \frac{1 - \mathrm{Wq0}}{m(1-\rho) - 1}\left(1 - e^{-(m\mu - \lambda) r}\right);$$

```
jain - gross
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$$1 - e^{-\frac{a r}{b}} + \frac{\left(1 - e^{r\left(-\frac{a m}{b} + \frac{a}{t}\right)}\right)\left(\frac{b}{t}\right)^m}{\left(-1 + m\left(1 - \frac{b}{m t}\right)\right)\left(1 - \frac{b}{m t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{\left(1 - \frac{b}{m t}\right) m!} + \frac{e^{b/t}\,\mathrm{Gamma}\left[m, \frac{b}{t}\right]}{\mathrm{Gamma}[m]}\right)} - \frac{\left(-e^{-\frac{a r}{b}} + e^{-\frac{a m r\left(1 - \frac{b}{m t}\right)}{b}}\right)\left(\frac{b}{t}\right)^m}{\left(1 - m + \frac{b}{t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{m!} + \frac{e^{b/t}\left(1 - \frac{b}{m t}\right)\mathrm{Gamma}\left[m, \frac{b}{t}\right]}{\mathrm{Gamma}[m]}\right)} - \frac{\left(1 - e^{-\frac{a r}{b}}\right)\left(-1 + m\left(1 - \frac{b}{m t}\right) + \frac{\left(\frac{b}{t}\right)^m}{\left(1 - \frac{b}{m t}\right) m!\left(\frac{\left(\frac{b}{t}\right)^m}{\left(1 - \frac{b}{m t}\right) m!} + \frac{e^{b/t}\,\mathrm{Gamma}\left[m, \frac{b}{t}\right]}{\mathrm{Gamma}[m]}\right)}\right)}{-1 + m\left(1 - \frac{b}{m t}\right)}$$

```
Simplify[%]
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0
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